The test particle motion equations metrical form in a potential field

K.B. Korotchenko

Russian Physics Journal, Vol. ??, No. 12, 1983

Abstract

It is shown in the present work that the three-dimensional trajectories of an electrical test particle in potential fields may be regarded as geodesic lines lying on isotropic surfaces of some four-dimensional configurational space, the connection of which has tortion, while the transference is nonmetric.

The starting point in the present work is the Einstein concept that the geometrization of an interaction consists in finding a metric space in which the test particles trajectories are geodesic lines [1].

An interesting method of metrization of arbitrary force interactions corresponding (to a definite extent) to this concept was proposed in [2]. In this method of metrization, the test particles move along geodesic lines. However, the force fields are related with the components of the connection tortion tensor of a pseudo-Euclidean space. In this sense, the metrization of force interactions proposed in [2] does not correspond to the Einstein program, since the metric properties of the space in which the force fields act do not depend on these force fields.

However, it is also of interest to consider a metric formulation of force interactions in which, as in [2], the test particles motion equations represent a special form of Newton's second law in four-dimensional form (constituting here the geodesic equation of some four-dimensional space ${}^kV_4$) but the metric tensor and physical fields are interdependent.

The construction of such a special geometric formulation of force interactions is the subject of the present work. To avoid problems associated with the distinction between the concepts of a reference frame and a coordinate system [2], different observers (i.e., reference frames) will be posed in accordance, generally speaking, with different four-dimensional spaces ${}^kV_4$.

To simplify the mathematical formalism, the test particles motion only in potential fields is considered in the present work.

1. The states of the test particles (of mass m and charge e) in the potential fields will be called classical states. Correspondingly, all the characteristics of the particle describing its behavior in the classical state (trajectory, velocity, momentum, energy, etc.) will be called classical.

It should be emphasized that below all classical characteristics are assumed to be specified with respect to one definite reference frame, which may be chosen in the form of any inertial frame (IF).

2. Suppose that ${}^kV_4$ is a four-dimensional space with the metric

$$ {}^{(k)}dS^2 \;=\; {}^{(k)}g_{oo}(x^i,t)c^2dt^2 \;+\; g_{ik}dx^idx^k \,, \tag{1}$$

where $(-g_{ik})$ is the metric tensor of the space V_3 (the "spatial" component of the Minkowski four-space V_4).

It is clear that any classical trajectory $x^i = x^i(t)$ may be regarded as a line on the isotropic surface ${}^kG_{o3} \subset {}^kV_4$, defined by the equation ${}^{(k)}g_{oo}c^2dt^2 = -g_{ik}dx^idx^k$, under the condition that along this line

$$ {}^{(k)}g_{oo}(x^i(t),t) \;=\; v^iv_i/c^2 \,, \tag{2}$$

where v^i is the particle velocity measured from the specified IF ($v_i = -g_{ik}v^k$).

Thus, each point p of the classical particle trajectory $x^i = x^i(t)$ in V_3 may be regarded as lying on the isotropic surface ${}^kG_{o3} \subset {}^kV_4$.

It is understood that in V_3 classical states of the particle are possible such that, at any point $p \in V_3$, the particle velocity is independent on the trajectory along which the particle reaches this point (of course, with certain specified initial parameters). For such states, each point $p \in V_3$ may also be regarded as a point of the isotropic surface ${}^kG_{o3}$ in ${}^kV_4$. That is an isotropic surface ${}^kG_{o3} \subset {}^kV_4$ may be constructed at such points of space V_3. By changing the values of the initial parameters, a set of surfaces ${}^kG_{o3}$ covering the whole of ${}^kV_4$ may be obtained. That is an imbedding [1] - enclosure in a space of higher dimensionality - may be constructed.

3. According to Eq.(1), the method of enclosure described in Sec.2 should have the distinctive property that the geometry of the enclosing space ${}^kV_4$ should have no influence on the geometric properties of the enclosed space V_3 (should not change the metric tensor g_{ik}). In other words, the imbedding must occur at those points of ${}^kV_4$ at which the external curvature of the enclosed surface is zero.

Then it follows from the GaussVaingarten equations - see [3], for example - that, with this imbedding, the absolute differential of the space ${}^kV_4$ (denoted by ${}^{(k)}\nabla(\cdots)$) is defined by the equation

$$ {}^{(k)}\nabla A^\mu \;=\; ({}^{(3)}\nabla_i A^\mu)dx^i \;+\; ({}^{(4)}\nabla_o A^\mu)dx^o \,. \tag{3}$$

Equation (3) may also be rewritten in the form

$$ {}^{(k)}\nabla A^{\mu} \;=\; {}^{(k)}DA^{\mu} \;+\; {}^{(k)}\Gamma^{\mu}_{\nu o} A^{\nu} dx^{o} \,, \tag{4} $$

where ${}^{(k)}DA^{i} = DA^{i} + {}^{(k)}S^{i}_{kl} A^{k} dx^{l}$ is the absolute differential of the pseudo-Euclidean space V_3 (here ${}^{(k)}S^{i}_{kl} = S_{kl}{}^{i} - S_{l}{}^{i}{}_{k} - S_{k}{}^{i}{}_{l}$ and $S_{kl}{}^{i}$ is the tortion tensor) and ${}^{(k)}\Gamma^{\mu}_{\nu o}$ is the connection of the space ${}^{k}V_4$.

4. To obtain a more detailed description, the definition of the absolute differential ${}^{(k)}\nabla(\cdots)$ is written in standard form [1], [4]

$$ {}^{(k)}\nabla A^{\mu} \;=\; (\partial_{\nu} A^{\mu} \;+\; \Gamma^{\mu}_{\omega\nu} A^{\omega}) dx^{\nu} \,, \tag{5} $$

where $2\Gamma^{\mu}_{\omega\nu} = 2{}^{(k)}\Gamma^{\mu}_{\omega\nu} + Q^{\mu}_{\omega\nu}$, at that $Q^{\mu}_{\omega\nu} = {}^{(k)}g^{\mu\gamma}({}^{(k)}Q_{\omega\nu\gamma} + {}^{(k)}Q_{\nu\gamma\omega} - {}^{(k)}Q_{\gamma\omega\nu})$ and ${}^{(k)}Q_{\mu\nu\omega} = -{}^{(k)}\nabla_{\mu}\left({}^{(k)}g_{\nu\omega}\right)$; ${}^{(k)}\Gamma^{\mu}_{\omega\nu} = \left\{ {\mu \atop \omega\nu} \right\} + {}^{(k)}S^{\mu}_{\omega\nu}$, where $\left\{ {\mu \atop \omega\nu} \right\}$ is the Christoffel symbol and ${}^{(k)}S^{\mu}_{\omega\nu} = S_{\omega\nu}{}^{\mu} - S_{\nu}{}^{\mu}{}_{\omega} - S_{\omega}{}^{\mu}{}_{\nu}$ at that $S_{\omega\nu}{}^{\mu}$ is the tortion tensor.

If it is required that the definition in Eq.(5) coincide with chat in Eq.(4), the result obtained is

$$ 2{}^{(k)}\Gamma^{i}_{oj} dx^{j} \;+\; Q^{i}_{o\omega} dx^{\omega} \;=\; Q^{i}_{j\omega} dx^{\omega} \;=\; 2{}^{(k)}\Gamma^{o}_{\mu j} dx^{j} \;+\; Q^{o}_{\mu\omega} dx^{\omega} \;=\; 0 \,, \tag{6} $$

which must be satisfied if the imbedding described in Secs.2 and 3 is to be possible.

It may readily be demonstrated that the absolute differential ${}^{(k)}\nabla(\cdots)$ of space ${}^{k}V_4$ defined by Eq.(4) describes a nonmetric transfer in ${}^{k}V_4$. In fact

$$ {}^{(k)}Q_{ooo} \;=\; 2{}^{(k)}g_{oo}{}^{(k)}S^{o}_{oo} \,, \quad {}^{(k)}Q_{ioo} \;=\; -\partial_{i}{}^{(k)}g_{oo} \,. \tag{7} $$

The remaining ${}^{(k)}Q_{\mu\nu\omega} = 0$. As a result equations (6) takes the form

$$ \begin{aligned} {}^{(k)}S^{o}_{oj} dx^{j} &= -\left\{ {o \atop oj} \right\} dx^{j} - 2{}^{(k)}S^{o}_{oo} dx^{o} \,, \\ {}^{(k)}S^{o}_{ij} dx^{j} &= -\left\{ {o \atop ij} \right\} dx^{j} + \left\{ {o \atop io} \right\} dx^{o} \,, \\ {}^{(k)}S^{i}_{oj} dx^{j} &= -\left\{ {i \atop oj} \right\} dx^{j} + \left\{ {i \atop oo} \right\} dx^{o} \,, \end{aligned} \tag{8} $$

Hence it is clear that the tortion $S_{\omega\nu}{}^{\mu}$ is nonzero.

Thus, the imbedding described in Sec.2 generates in ${}^{k}V_4$ a geometry with tortion and a nonzero covariant derivative of the metric tensor.

5. The test particle motion equations are now considered. It is desirable for these equations to coincide with the geodesic equations in ${}^{k}V_4$. Then these equations should take the form

$$ Dp^{\mu} \;=\; -{}^{(k)}\Gamma^{\mu}_{\nu o} p^{o} dx^{\nu} \,, \quad (p^{\mu} \;=\; m dx^{\mu}/d\tau) \,. \tag{9} $$

Taking this into account, the condition $dx_i dp^i = dx^o dp^o$ leads to the equation

$$^{(k)}S^j_{\nu o} dx^\nu dx_j \;=\; \left(\left\{{o \atop \nu o}\right\} + {}^{(k)}S^o_{\nu o}\right) dx^\nu dx^o - \left\{{j \atop \nu o}\right\} dx^\nu dx_j \;, \tag{10}$$

which, together with Eq.(8), describes all the nonzero components of $^{(k)}S^\mu_{\omega\nu}$.

Further, it is readily evident that, if the components $^{(k)}S^o_{\nu o}$ are chosen in the form

$$^{(k)}S^o_{\nu o} \;=\; [\partial_\nu \ln((1 - {}^{(k)}g_{oo})/{}^{(k)}g_{oo})]/2 \;, \tag{11}$$

the four-momentum p^o component is found to be

$$p^o \;=\; C_1(1 - {}^{(k)}g_{oo})^{-1/2} \;, \tag{12}$$

where $C_1 = const.$ Assuming that $C_1 = mc$, it is found that $d\tau = (1 - {}^{(k)}g_{oo})^{-1/2} dt$.

Hence, Eqs.(10) and (11) are the necessary and sufficient conditions for the motion equations Eq.(9) to be noncontradictory.

Thus, the classical particle trajectories in the potential fields specified with respect to a definite IF may be represented as geodesic lines lying on isotropic surfaces of some configurational space $^kV_4$ the connection of which has tortion, while the transference is nonmetric. The geometry of the space $^kV_4$ has the distinctive property that the magnitude of the nonmetricity of the transfer and the tortion are determined by specifying the metric coefficient $^{(k)}g_{oo}$ under the condition that the mixed components $^{(k)}g_{oi} \equiv 0$.

In conclusion, thanks are due to G. I. Flesher for interest in the work and for useful critical comments and also to I. L. Bukhbinder for fruitful discussions of the fundamental aspects of the present work.

Referenses

[1] I. P. Konopleva and V. N. Popov, Gauge Fields [in Russian], Atomizdat, Moscow (1980).

[2] V. I. Rodichev, Theory of Gravitation in an Orthogonal Reference Frame [in Russian], Atomizdat, Moscow (1980).

[3] C. W. Misner et al.. Gravitation, Vol. 2, Cambridge University Press, New York (1973).

[4] A. Likhnerovich, Theory of Connectedness as a Whole and Holonomic Groups [Russian translation], IL, Moscow (1960).